Propositional logic: Semantics

CS242 Formal Specification and Verification

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Semantics

Provability:

$$\phi_1, \phi_2, \ldots, \phi_n \vdash \psi$$

asserts that, from premises ϕ_1, ϕ_2, ..., ϕ_n, we can prove ψ in natural deduction.

Semantic entailment:

$$\phi_1, \phi_2, \ldots, \phi_n \models \psi$$

asserts that, for every assignment of `T` or `F` to all atomic propositions, if each of ϕ_1, ϕ_2, ..., ϕ_n evaluates to `T`, then ψ also evaluates to `T`.

Truth tables

ϕ	ψ	$\phi \wedge \psi$
T	T	T
T	F	F
F	T	F
F	F	F

ϕ	ψ	$\phi \vee \psi$
T	T	T
T	F	T
F	T	T
F	F	F

ϕ	ψ	$\phi \rightarrow \psi$
T	T	T
T	F	F
F	T	T
F	F	T

ϕ	$\neg\phi$
T	F
F	T

$\top$
T

$\bot$
F

Example truth tables:

$$(p \to \neg q) \to (q \vee \neg p)$$

$$((p \to q) \to p) \to p$$

Ordinary induction

If we have:

base case: $M(1)$ holds;

inductive step: $M(n)$ holds for a natural number $n \geq 1$ (*inductive hypothesis*) implies that $M(n+1)$ holds;

then we can conclude that $M(n)$ holds for every natural number $n \geq 1$.

Example proof:

$$1 + 2 + \cdots + n = \frac{n \cdot (n+1)}{2}$$

Course-of-values induction

If we have:

inductive step: $M(1)$, $M(2)$, ..., $M(n)$ all hold for a natural number $n \geq 0$ (*inductive hypothesis*) implies that $M(n+1)$ holds;

then we can conclude that $M(n)$ holds for every natural number $n \geq 1$.

Structural induction

The *height* of a well-formed formula ϕ is 1 plus the length of the longest path of its parse tree.

Structural induction is course-of-values induction on the height of well-formed formulas.

Example proof:
For every well-formed propositional logic formula, the number of left brackets is equal to the number of right brackets.

Soundness

$$\phi_1, \phi_2, \ldots, \phi_n \vdash \psi$$

implies

$$\phi_1, \phi_2, \ldots, \phi_n \models \psi$$

Some examples:

$$p \wedge q \models p$$

$$p \to q \models q$$

$$p \to q, \neg q, p \models \bot$$

$$p \models p \wedge q$$

Proof of soundness

For all natural numbers $k \geq 1$, we have that for all sequents

$$\phi_1, \phi_2, \ldots, \phi_n \vdash \psi$$

which have a proof of length k, it is the case that

$$\phi_1, \phi_2, \ldots, \phi_n \models \psi$$

We prove this by course-of-values induction on k.

Completeness

$$\phi_1, \phi_2, \ldots, \phi_n \models \psi$$

implies

$$\phi_1, \phi_2, \ldots, \phi_n \vdash \psi$$

Proof:

Step 1: $\models \phi_1 \rightarrow (\phi_2 \rightarrow (\cdots \rightarrow (\phi_n \rightarrow \psi) \cdots))$

Step 2: $\vdash \phi_1 \rightarrow (\phi_2 \rightarrow (\cdots \rightarrow (\phi_n \rightarrow \psi) \cdots))$

Step 3: $\phi_1, \phi_2, \ldots, \phi_n \vdash \psi$

Step 2

$\models \eta$ implies $\vdash \eta$.

Proposition

Let ϕ be a formula such that $p_1, p_2, \ldots, p_m$ are its only propositional atoms. Let l be any line number in ϕ's truth table. For all i, let

$$\hat{p}_i^l = \begin{cases} p_i, & \text{if the entry for } p_i \text{ in line } l \text{ is } \mathtt{T} \\ \neg p_i, & \text{if the entry for } p_i \text{ in line } l \text{ is } \mathtt{F} \end{cases}$$

Then we have

1. $\hat{p}_1^l, \hat{p}_2^l, \ldots, \hat{p}_m^l \vdash \phi$ is provable if the entry for ϕ in line l is $\mathtt{T}$;
2. $\hat{p}_1^l, \hat{p}_2^l, \ldots, \hat{p}_m^l \vdash \neg\phi$ is provable if the entry for ϕ in line l is $\mathtt{F}$.

Proof

By structural induction on ϕ.

It remains to put together proofs of the sequents

$$\hat{p}_1^l, \hat{p}_2^l, \ldots, \hat{p}_m^l \vdash \eta$$

for $l \in \{1, \ldots, 2^m\}$.
Example:

$$\eta = p \to (q \to p)$$